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Phase transitions for a rock–scissors–paper model with long-range-directed interactions

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Abstract
We have investigated a rock–scissors–paper model with long-range-directed interactions in two dimensions where every site has four outgoing links but a fraction q of the outgoing links to the nearest neighbour sites are rewired to other long-distance sites chosen randomly and the lattice structure is replaced again after a Monte Carlo step. It is found that, with q increasing, the system changes from a three species coexistence self-organizing state to a global oscillation state and then to one of the homogeneous states. However when q exceeds a third threshold value, the system returns to a self-organizing state. When we restrict the maximum number of ingoing links of a site to four, the last self-organizing state disappears, the system stays in the homogeneous state forever after q exceeds the second threshold value. And when we restrict the maximum number of ingoing links of a site to five or six, the system exhibits a transition from the homogeneous state to a global oscillation state again and then to the last self-organizing state with q increasing. The comparison of results on different networks suggests that the sites with zero ingoing links should play a significant role in the emergences of the later self-organizing state and the subsequent global oscillation.

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1. Introduction

Recently, dynamical behaviours defined on various complex networks have been extensively studied in many fields, such as physics, biology and even society science including economics and epidemic spreading [1–4]. In these investigations, the traditional models in the realm of physics, such as the Ising model and XY model [6], have been studied in networks to understand the complex dynamical behaviours occurring in a real world. Because some important features

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of a real system such as short path length and high cluster can be well modelled by small-world networks (SWN) [7] such as the known Watts–Strogatz (WS) network, many dynamical processes have been investigated on WS-like networks [5, 8].

Szabó and his collaborators [9, 10] have studied the dynamics of a spatial game model [11, 12] of rock–scissors–paper (RSP) on a small-world network in which a portion q of links between the nearest neighbour sites located on a regular lattice are substituted by links to a long-distance sites randomly selected. The spatial RSP model is a three species predator–prey model where the species dominate each other cyclically, the time evolution of this system is governed by the iteration of cyclic invasion processes between two randomly chosen neighbouring sites, namely, the pair (1, 2) or (2,1) changes to (1, 1), (2, 3) or (3, 2) changes to (2, 2), and (3, 1) or (1, 3) becomes (3, 3) with a same unit probability. They have observed that on a quenched structure a limit cycle occurs when the portion q exceeds a threshold value. On an annealed network structure this transition can also be observed, however, when q exceeds a second threshold value, the system enters into one of the homogeneous states (an absorbing state).

The above RSP model has a crucial feature: the links between two sites are symmetric, if A is linked to B, then B must also be linked to A. However, in a real world, many other networks are definitely asymmetric and their links are directed as, for instance, in the case of networks of the import and the export of goods, food webs, World Wide Web page links, etc. It has been found that the character of directed ties has an important influence on the dynamical behaviour in real networks [1, 13–21]. We think that an asymmetric network is very adapt to the predator–prey model because a predator cannot always capture the preys when they meet together. In a real world, a predator has to take a suitable position as possible to capture a prey or a prey is on a dangerous position where the prey can be captured easily, otherwise almost every kind of prey has an ability to escape from the predator.

In this paper, we have investigated the RSP model on directed small-world networks where every site has four outgoing links but a fraction q of the outgoing links to the nearest neighbour site are rewired to another long-distance sites randomly chosen [13].

To construct a directed small-world network, we start with a two-dimensional ($L \times L$) square lattice with a periodic boundary condition, every site has four outgoing and four ingoing links to their nearest neighbours, respectively. We rewire a nearest neighbour outgoing link with a probability of q ($0 \leqslant q \leqslant 1$) to another different site selected randomly while self-connected link is forbidden. Then we have a small-world network with directed links after repeating the rewiring process over all of nearest neighbour outgoing links. Then the four sites linked by the four outgoing links are the mates instead of the four nearest neighbour sites. We can see that by this procedure every site will have exactly four outgoing links and a varying number of incoming links. In our simulations, the neighbouring sites are not fixed, we change the directed small-world structure per Monte Carlo step (MCS) (a MCS means $L \times L$ Monte Carlo attempts).

In the simulation, the individual located on site i of a lattice belongs to one of the three species ($S_i = 1, 2, 3$) which dominate each other cyclically ($1 \to 2 \to 3 \to 1$). We randomly choose a pair of sites which are linked by a directed outgoing link from one site to another. Owing to the directed link, the evolutionary dynamical rule is directed, for example, a pair (1, 2) is changed into (1, 1) but do nothing for a pair (2, 1).

From the above definition, we can see that our model has two ingredients to describe a kind of spatial heterogeneity which exists extensively in a predator–prey ecosystem and has attracted much attention of ecologists [22]: the link between a predator and a prey is asymmetric and the number of the incoming links to a site is varied. It is very interesting to study the effect of this kind of spatial heterogeneity on the evolutional behaviour.

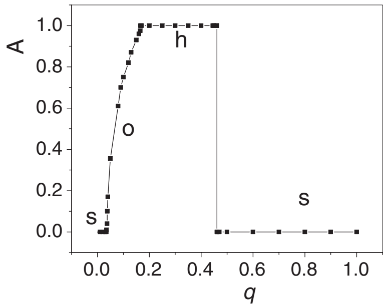

Figure 1. Simulation results for RSP model on an annealed directed small-world network. The maximum number of ingoing links of a site is not restricted. Here A is defined as the amplitude of global oscillation state, s is defined as the self-organizing state, o is defined as the global oscillation state and h is defined as the homogeneous state. For $A = 0$ the system is in a self-organizing state where three species coexist; meanwhile for $A = 1$ the system is in one of the homogeneous states.

2. Simulation result

A Monte Carlo simulation is started from a random initial state where the three species take their place with a same probability (1/3) on a square lattice (1024×1024). The time evolution is governed by invasions between neighbours with a probability $1 - q$ or along a long-range link chosen with a probability q. When $q = 0$, this system evolves into a stationary state where all three species are present with the same average concentration; in this situation, the three species alternate cyclically at each site and the short-range interactions are not able to synchronize these local oscillations. This self-organizing state can also be observed for weak randomness ($q < q_1 = 0.034 \pm 0.004$).

When q exceeds q_1, a global oscillation state occurs, and the amplitude A of the global oscillation increases monotonically with q as shown in figure 1. This transition is a Hopf bifurcation which has been well studied by mean-field-type approaches [23]. The same phenomenon was observed by Kuperman and Abramson [24], they have supposed that the emergence of the global oscillation state should be related to the variation of the clustering coefficient. But Szolnoki and Szabó [9] have suggested that the clustering coefficient could not play a significant role in this phase transition. We think that when q is below the threshold value, the system evolves into a domain structure, the short-range interactions cannot synchronize the local oscillations. But when q exceeds the threshold value, the long-range interactions destroy these domains and then synchronize the local oscillations.

When q exceeds a second threshold value $q_2 = 0.165 \pm 0.005$, as shown in figure 1, the system sooner or later evolves into one of the homogeneous states containing only one specie. Evidently, this absorbing phase is due to the annealed structure. The same phenomenon was observed by Szolnoki and Szabó [9]. When they use a quenched structure, the amplitude of oscillation tends to a fixed value in the limit $q \rightarrow 1$, and when they use an annealed structure the amplitude of the oscillation increases with q increasing and finally the evolution ends in one of the homogeneous states. However, it is surprised that when q exceeds a third threshold value $q_3 = 0.461 \pm 0.004$, the system returns to a self-organizing state again in our simulation.

In the above simulation we do not restrict the number of ingoing links of a site. Every site has four outgoing links, but the ingoing links are not fixed, therefore ingoing links of some sites may be zero. From the result shown in figure 2, we can see that the number of sites with zero ingoing link increases with q increasing. It can be conjectured that the emergence of the

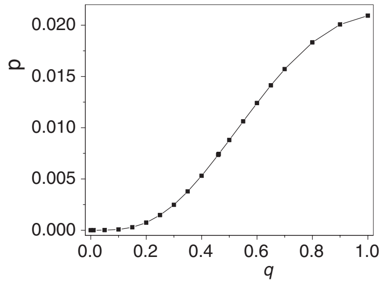

Figure 2. The change of the number of the zero ingoing link of sites with the increase in q. Here p is defined as the percentage of the sites with zero ingoing link in all sites.

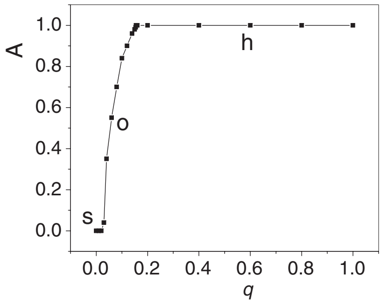

Figure 3. Simulation results for RSP model on an annealed directed small-world network. The maximum number of ingoing links of a site is restricted to 4.

last self-organizing state is related to the number of sites with zero ingoing link. When q is very small, the sites with zero ingoing link cannot affect the system seriously. But with the increase in the number of the sites with zero ingoing link, more and more preys on these sites are protected from the predators, therefore, when q exceeds a certain value, sites with zero ingoing link may affect the dynamical behaviour of the system seriously and lead to that the system returns to the self-organizing state abruptly again.

In contrast, when we restrict the maximum number of ingoing links of a site to 4, the phase transitions are clearly shown in figure 3, we can find that when q exceeds a threshold value $q_1 = 0.02 \pm 0.004$ the system changes from a self-organizing state to a global oscillation state, and when q exceeds a second threshold value $q_2 = 0.15 \pm 0.005$ the system stays in one of the homogeneous states for ever. In this situation, every site has four ingoing links and four outgoing links, there is no site with zero ingoing link. These simulation results are in consistence with the previous observation [9] in an annealed structure.

In order to have a deep understanding of the effect of the structure, we restrict the maximum number of ingoing links of a site to 5 and 6. When the maximum number of ingoing links is restricted to 5, we can also find that the system changes from a self-organizing state to a global oscillation state and then to a homogeneous state with q increasing. However, as shown in figure 4, when q exceeds a third threshold value $q_3 = 0.66 \pm 0.004$ the system returns to a global oscillation state, and when q exceeds a fourth threshold value $q_4 = 0.93 \pm 0.005$ the system finally ends in a self-organizing state. The same phenomena can be found in figure 5 when we restrict the maximum number of ingoing links of a site to 6, at this case the third threshold value is $q_3 = 0.56 \pm 0.005$ and the fourth threshold value is $q_4 = 0.62 \pm 0.005$.

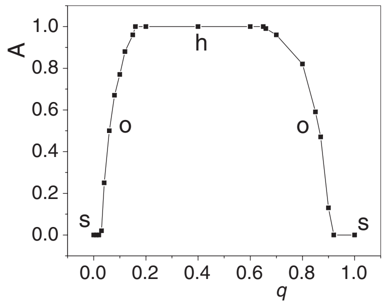

Figure 4. Simulation results for RSP model on an annealed directed small-world network. The maximum number of ingoing links of a site is restricted to 5.

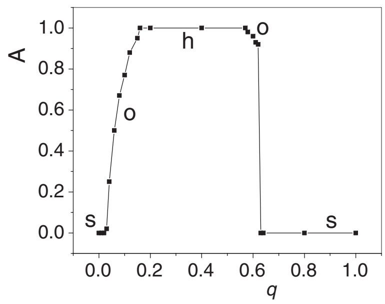

Figure 5. Simulation results for RSP model on an annealed directed small-world network. The maximum number of ingoing links of a site is restricted to 6.

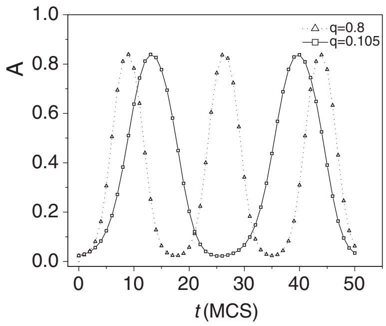

Figure 6. The contrast about the period of the global oscillation state between $q = 0.105$ and $q = 0.8$ when the maximum number of ingoing links of a site is restricted to 5.

We find that the subsequent global oscillation state has some differences with the first one, the relaxation time to the later global oscillation state from an initial state is much longer than that of the first one, but the period of the later global oscillation state is shorter than that of the first global oscillation state under the same amplitude oscillation as shown in figure 6.

When the maximum number of permitted ingoing links of a site is 7, the dynamical evolution is returned to the same simulation results as shown in figure 1 because the number of the sites with seven ingoing links is very small.

3. Conclusion

To summarize, we have investigated the dynamical behaviour of RSP model on two-dimensional annealed directed small-world networks. In our simulation, we change the directed small-world structure per Monte Carlo step. From simulation results we have found that, with the increase in q, the system changes from a self-organizing state to a global oscillation state and then to one of the three homogeneous states. However, when q exceeds a third threshold value, the system returns to a self-organizing state. When we restrict the maximum number of ingoing links of a site to 4, the last self-organizing state disappears, the system stays in the homogeneous state forever after q exceeds a second threshold value like the previous model [9, 10]. And when we restrict the maximum number of ingoing links of a site to 5 or 6, the system exhibits a transition from the homogeneous states to a global oscillation state again and then to the last self-organizing state with q increasing. We can conjecture that the number of the sites with zero ingoing link takes an important role in the formations of the later self-organizing state and the subsequent global oscillation. From our simulation results, we can find that a spatial heterogeneity can have an important effect on the evolutional behaviour of a predator–prey system.

Acknowledgment

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